Matrices and Probability

Vectors and Matrices

A **vector** is an expression of the form:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$$

Vectors and Matrices

Examples of vectors:

$$\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 1.5 & 2.4 & 0.7 & 4.2 \end{pmatrix}$$

column vector

row vector

Vectors and Matrices

Zero vectors:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$$

Vectors and Matrices

A **matrix** is an expression of the form:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Vectors and Matrices

Examples of matrices:

$$\begin{pmatrix} 2 & 0 \\ -5 & 3 \end{pmatrix} \quad , \quad \begin{pmatrix} 4.3 & 2.7 \\ 0.5 & 1.3 \\ 2.3 & 3.8 \end{pmatrix}$$

2×2 matrix $\qquad$ 3×2 matrix

Vectors and Matrices

no. of rows

no. of columns

$m \times n$ matrix

a_{ij} is the entry in the
i^{th} row and j^{th} column

Vectors and Matrices

Example:

$$[a_{ij}] = \begin{pmatrix} 4.3 & 2.7 \\ 0.5 & 1.3 \\ 2.3 & 3.8 \end{pmatrix}$$

is a 3×2 matrix.

$$a_{21} = 0.5, \quad a_{32} = 3.8$$

Transpose

The transpose of a matrix $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

is

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

Transpose

If v is a row vector, then v^T is a column vector and vice versa.

Example: If

$$v = \begin{pmatrix} 4 & -1 & 2 \end{pmatrix}$$

then

$$v^T = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$$

Transpose

If A is an $m\times n$ matrix, then A is an $n\times m$ matrix.

Example: If

$$A=\begin{pmatrix} 2 & -3 \\ -1 & 0 \\ 4 & 1 \end{pmatrix}$$

then

$$A^T=\begin{pmatrix} 2 & -1 & 4 \\ -3 & 0 & 1 \end{pmatrix}$$

Matrix addition

Sum of two matrices is defined by

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

In other words,

$$[a_{ij}]+[b_{ij}]=[a_{ij}+b_{ij}]$$

Matrix addition

Examples:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 \\ 1 & 4 \\ 2 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 4 & 8 \\ 7 & 11 \end{pmatrix}$$

Scalar Multiplication

Scalar multiplication is defined by

$$k\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

In other words,

$$k[a_{ij}] = [ka_{ij}]$$

Scalar Multiplication

Examples:

$$-2 \cdot \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix}$$

$$3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \\ 15 & 18 \end{pmatrix}$$

Inner product of vectors

Let $\mathbf{u}=(u_1,u_2,\cdots,u_n)$ be a row vector
and $\mathbf{v}=(v_1,v_2,\cdots,v_n)^T$ be a column vector
then we may define the product

$$\begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

Note that the product **uv** is a real number (scalar).

Inner-product of vectors

Example:

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6$$

$$= 32$$

Dim-sum Prices

Type	Special	Large	Medium	Small
Prices	$20	$15	$10	$8

Price vector:

$$\begin{pmatrix} 20 & 15 & 10 & 8 \end{pmatrix}$$

Dim-sum Prices

Type	Special	Large	Medium	Small
Quantity	5	8	4	6

Consumption vector:

$$\begin{pmatrix} 5 & 8 & 4 & 6 \end{pmatrix}^T$$

Dim-sum Prices

You need to pay:

$$\begin{pmatrix} 20 & 15 & 10 & 8 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \\ 4 \\ 6 \end{pmatrix}$$

$$= 20 \times 5 + 15 \times 8 + 10 \times 4 + 8 \times 6$$

$$= 308$$

Matrix Multiplication

Let $A=[a_{ik}]$ be an m-by-p matrix and $B=[b_{kj}]$ be a p-by-n matrix. The matrix product AB is defined as an m-by-n matrix such that the ij-th entry $[AB]_{ij}$ of AB is the product of the i-th row vector of A and the j-th column vector of B. In other words

$$[AB]_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}$$

Matrix Multiplication

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{pmatrix} = \begin{pmatrix} 1\times 0.1 + 2\times 0.2 & 1\times 0.3 + 2\times 0.4 \\ 3\times 0.1 + 4\times 0.2 & 3\times 0.3 + 4\times 0.4 \\ 5\times 0.1 + 6\times 0.2 & 5\times 0.3 + 6\times 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.5 & 1.1 \\ 1.1 & 2.5 \\ 1.7 & 3.9 \end{pmatrix}$$

Matrix Multiplication

no. of columns — no. of rows

$$m \times p \text{ matrix} \cdot p \times n \text{ matrix} = m \times n \text{ matrix}$$

Matrix Multiplication

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ip} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ & & & & \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{pmatrix}$$

$$= \begin{pmatrix} c_{11} & \cdots & c_{1j} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mj} & \cdots & c_{mn} \end{pmatrix}$$

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{ip} b_{pj}$$

Matrix Multiplication

Example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

Matrix Multiplication

Example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

Matrix Multiplication

Example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{pmatrix}$$

Matrix Multiplication

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1.1 & 1.2 & 1.3 \\ 2.1 & 2.2 & 2.3 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1.1 + 2\times 2.1 & 1\times 1.2 + 2\times 2.2 & 1\times 1.3 + 2\times 2.3 \\ 3\times 1.1 + 4\times 2.1 & 3\times 1.2 + 4\times 2.2 & 3\times 1.3 + 4\times 2.3 \\ 5\times 1.1 + 6\times 2.1 & 5\times 1.2 + 6\times 2.2 & 5\times 1.3 + 6\times 2.3 \end{pmatrix}$$

Matrix Multiplication

Matrix multiplication:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1.1 & 1.2 & 1.3 \\ 2.1 & 2.2 & 2.3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$a_{21} = \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 1.1 \\ 2.1 \end{pmatrix}$$

$$= 3 \times 1.1 + 4 \times 2.1$$

$$= 11.7$$

Matrix Multiplication

Matrix multiplication:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1.1 & 1.2 & 1.3 \\ 2.1 & 2.2 & 2.3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$a_{33} = \begin{pmatrix} 5 & 6 \end{pmatrix} \begin{pmatrix} 1.3 \\ 2.3 \end{pmatrix}$$

$$= 5 \times 1.3 + 6 \times 2.3$$

$$= 20.3$$

Dim-sum Prices

Restaurant	Special	Large	Medium	Small
A	$20	$15	$10	$8
B	$18	$14	$12	$10
C	$23	$13	$11	$7

Price matrix:

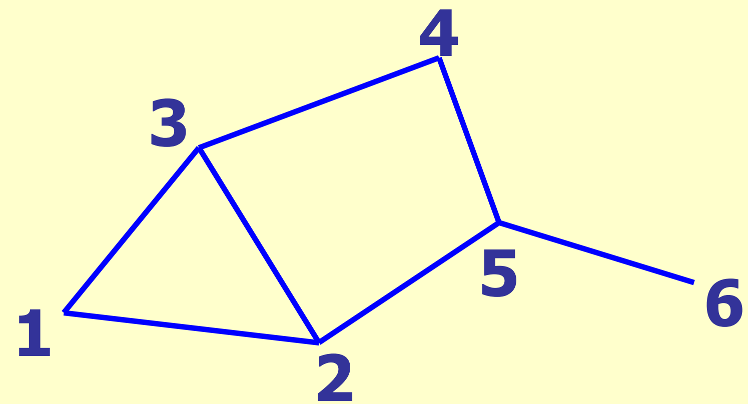
$$\begin{pmatrix} 20 & 15 & 10 & 8 \\ 18 & 14 & 12 & 10 \\ 23 & 13 & 11 & 7 \end{pmatrix}$$

Dim-sum Prices

$$\begin{pmatrix} 20 & 15 & 10 & 8 \\ 18 & 14 & 12 & 10 \\ 23 & 13 & 11 & 7 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 308 \\ 310 \\ 305 \end{pmatrix}$$

Restaurant	Cost
A	$308
B	$310
C	$305

Incident Matrix



Incident matrix:

$$I = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

'3' and '1' are jointed

'4' and '3' are jointed

'2' and '4' are not jointed

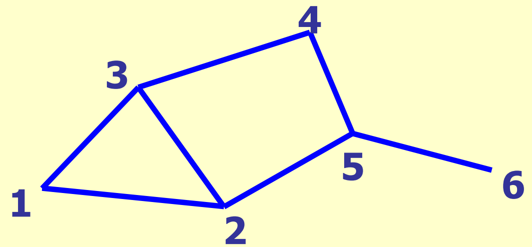
'5' and '5' are not jointed

Incident Matrix

$$I^2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Incident Matrix



$$I^2 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 1 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 & 2 & 0 \\ 1 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

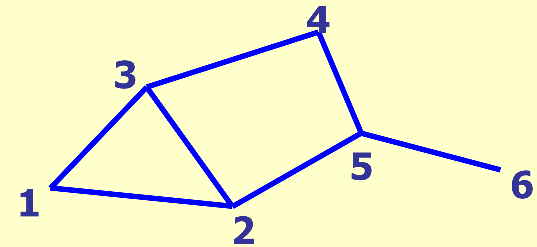
Degree of '2' is 3

There is 1 way to travel from '1' to '5' with 2 steps

There are 2 ways to travel from '3' to '5' with 2 steps

There is 0 way to travel from '2' to '5' with 2 steps

Incident Matrix



$$I^3 = \begin{pmatrix} 2 & 4 & 4 & 2 & 2 & 1 \\ 4 & 2 & 6 & 1 & 6 & 0 \\ 4 & 6 & 2 & 5 & 1 & 2 \\ 2 & 1 & 5 & 0 & 5 & 0 \\ 2 & 6 & 1 & 5 & 0 & 3 \\ 1 & 0 & 2 & 0 & 3 & 0 \end{pmatrix}$$

There are 6 ways to travel from '3' to '2' with 3 steps

There is 1 way to travel from '1' to '6' with 3 steps

There are 5 ways to travel from '5' to '4' with 3 steps

Probability

Definition: **A (finite) sample space is the set of all possible outcomes of a random trial.**

Examples:

1. **Throwing a dice** $S = \{1,2,3,4,5,6\}$
2. **Tossing two coins** $S = \{HH, HT, TH, TT\}$
3. **Drawing a poker card** $S =$ **{♠A,♥A,…,♦K}**

Probability

***Definition*: An event *A* is a collection of outcomes. ($A \subset S$)**

Examples: For throwing a dice,

1. **The outcome is six** $A = \{6\}$
2. **The outcome is even** $A = \{2,4,6\}$
3. **The outcome > 4** $A = \{5,6\}$

Probability

Definition: A probability function p is a function such that

$$1.\ 0 \le p(E) \le 1, \quad \textit{for any event } E \subset S.$$

$$2.\ p(E_1 \cup E_2) = p(E_1) + p(E_2),$$

$$\textit{when } E_1 \textit{ and } E_2 \textit{ are mutually exclusive.}$$

$$3.\ p(S) = 1$$

Monty Hall’s Problem

Monty Hall's Problem

Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2?" Is it to your advantage to switch your choice?

Monty Hall’s Problem

Monty Hall’s Problem

Monty Hall’s Problem

Monty Hall's Problem

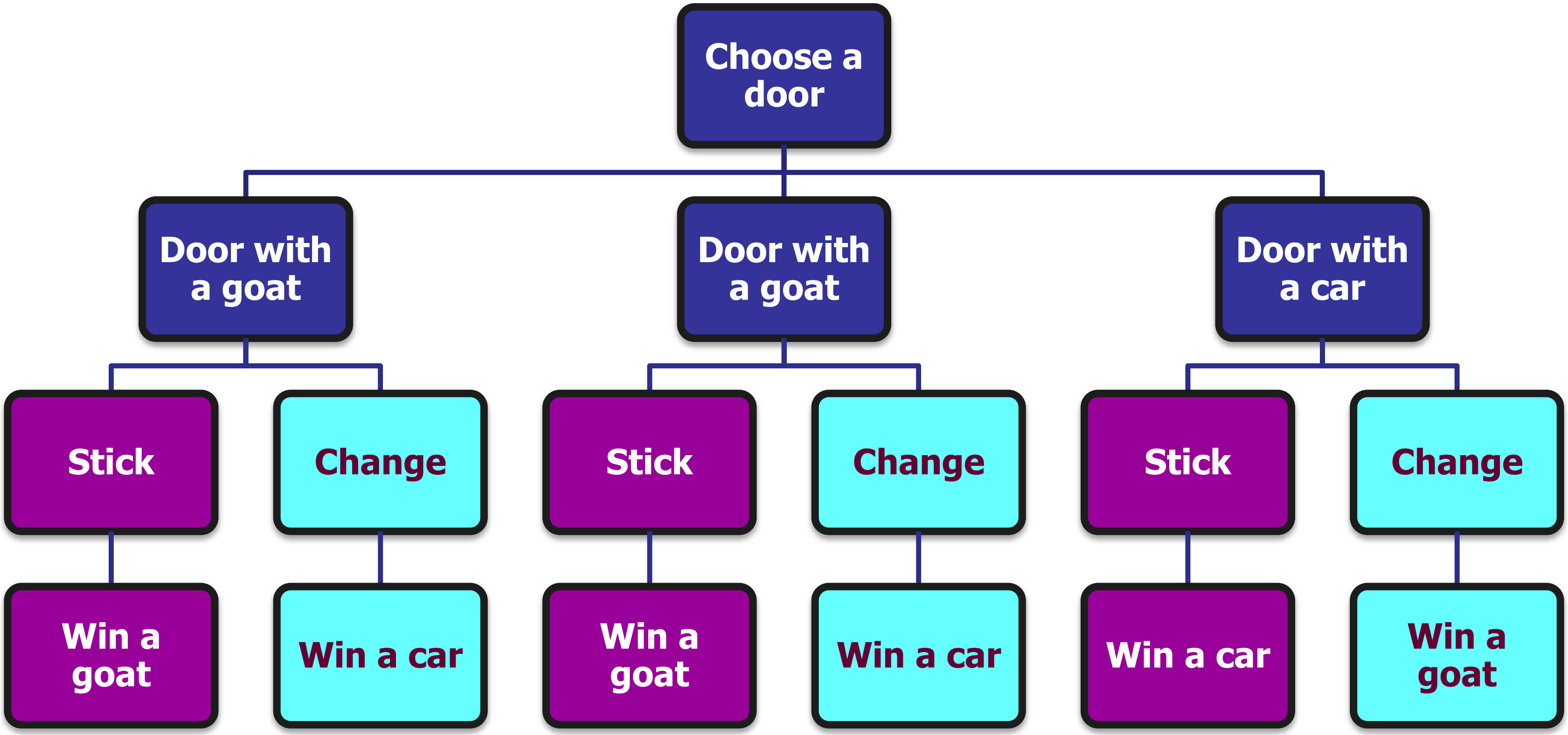
Stick or Change

Marilyn vos Savant

Ask Marilyn

Parade Magazine

Monty Hall's Problem

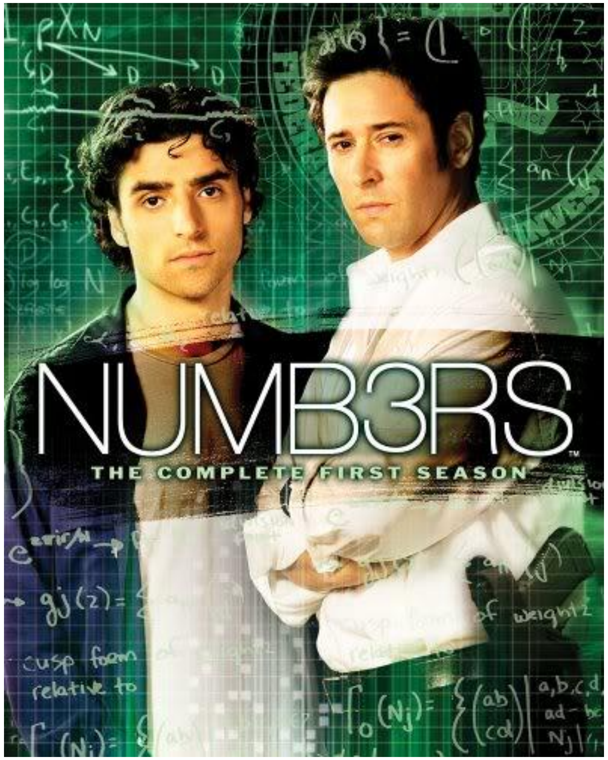


Monty Hall's Problem

Probability

Strategy	Win a Goat	Win a Car
Stick	2/3	1/3
Change	1/3	2/3

Monty Hall's Problem on Movies



http://www.youtube.com/watch?v=5e_NKJD7msg&feature=related
http://www.youtube.com/watch?v=mhlc7peGlGg

Random Variable

Definition: If we assign a value X to each element in a sample space, then we say that X is a random variable. (In other words, X is a function defined on a sample space S.)

Random Variable

A random variable usually has a practical meaning.

Examples:

1. **X = number shown on a dice.**
2. **X = sum of the numbers on two dice.**
3. **X = number of heads shown on three coins.**

Random Variable

You may also assign the values in any way you want.

Examples: For drawing a poker card, define

X(♠A) = 1, X(♥A) = 3, X(♣A) = 6, X(♦A) = 3,

X(♠2) = 4.7, X(♥2) = -1.9,…, X(♦K) = 13.

Expected Value

Definition: The expected value of a random variable X is defined as

$$E(X) = \sum xP(X = x)$$

Expected Value

Examples:

1. ***X* = number shown on a dice.**

$$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6} = 3.5$$

2. ***X* = sum of the numbers on two dice.**

$$E(X) = 2 \times \frac{1}{36} + 3 \times \frac{1}{18} + 4 \times \frac{1}{12} + \cdots + 12 \times \frac{1}{36} = 7$$

Expected Value

Examples:

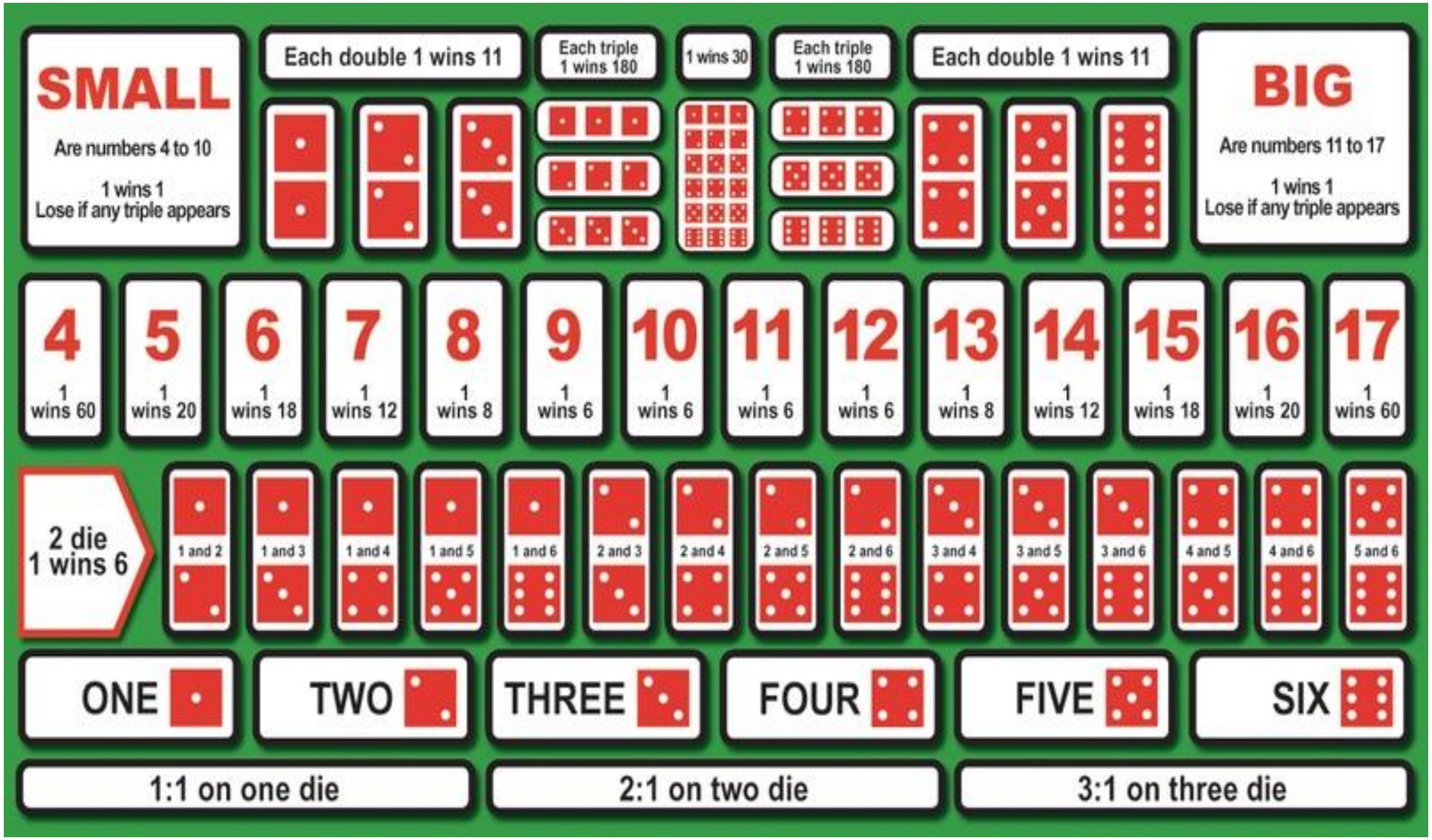
1. ***X*** **= number of heads shown on three coins.**

$$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

2. ***X*** **= Gain in a typical "Big or Small" game.**

$$E(X) = 1 \times \frac{35}{72} + (-1) \times \frac{37}{72} = -\frac{1}{36}$$

Sic Bo



SMALL
Are numbers 4 to 10
1 wins 1
Lose if any triple appears
Each double 1 wins 11
Each triple 1 wins 180
1 wins 30
Each triple 1 wins 180
Each double 1 wins 11
BIG
Are numbers 11 to 17
1 wins 1
Lose if any triple appears
4 1 wins 60
5 1 wins 20
6 1 wins 18
7 1 wins 12
8 1 wins 8
9 1 wins 6
10 1 wins 6
11 1 wins 6
12 1 wins 6
13 1 wins 8
14 1 wins 12
15 1 wins 18
16 1 wins 20
17 1 wins 60
2 die 1 wins 6
1 and 2
1 and 3
1 and 4
1 and 5
1 and 6
2 and 3
2 and 4
2 and 5
2 and 6
3 and 4
3 and 5
3 and 6
4 and 5
4 and 6
5 and 6
ONE
TWO
THREE
FOUR
FIVE
SIX
1:1 on one die
2:1 on two die
3:1 on three die

Expected Value

The expected payoff for

1. Big: $E(X)=1\times\frac{105}{216}+(-1)\times\frac{111}{216}=-\frac{6}{216}$

2. Triple 6: $E(X)=180\times\frac{1}{216}+(-1)\times\frac{215}{216}=-\frac{35}{216}$

3. 15 points: $E(X)=18\times\frac{10}{216}+(-1)\times\frac{206}{216}=-\frac{26}{216}$

4. No. of 1: $E(X)=3\times\frac{1}{216}+2\times\frac{15}{216}+1\times\frac{75}{216}+(-1)\times\frac{125}{216}=-\frac{17}{216}$

Matrix representation of Games

A two-person game with finite number of strategies can be represented by a matrix. Usually, the rows correspond to strategies of Player I and we say that Player I is the row player. Similarly, Player II is called the column player.

Matrix representation of Games

		Player II (Column Player)			
		C_1	C_2	…	C_n
Player I (Row Player)	R_1	(a_{11},b_{11})	(a_{12},b_{12})	…	(a_{1n},b_{1n})
	R_2	(a_{21},b_{21})	(a_{22},b_{22})	…	(a_{2n},b_{2n})
	…	…	…	…	…
	R_m	(a_{m1},b_{11})	(a_{m2},b_{m2})	…	(a_{mn},b_{mn})

Payoffs of the game

We may also use two matrices to represent the payoffs of the players.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Payoff matrix of Player I

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

Payoff matrix of Player II

Strategies of the Players

A mixed strategy of Player I is represented by a row vector

$$\mathbf{p} = \begin{pmatrix} p_1 & p_2 & \cdots & p_m \end{pmatrix}$$

It means that Player uses strategies R_1, R_2 ,…, R_m, with the probabilities p_1, p_2 ,…, p_m, respectively.

Note that we have:

$$0 \le p_k \le 1 \quad \textbf{and} \quad \sum_{k=1}^{m} p_k = 1$$

Strategies of the Players

Similarly, a mixed strategy of Player II is represented by another row vector

$$\mathbf{q} = \begin{pmatrix} q_1 & q_2 & \cdots & q_n \end{pmatrix}$$

where

$$0 \le q_l \le 1 \quad \textbf{and} \quad \sum_{l=1}^{n} q_l = 1$$

Expected Payoffs

Then the expected payoff of Player I is

$$
\begin{aligned}
&E(P_A)\\
&= a_{11}p_1q_1 + a_{12}p_1q_2 + \cdots + a_{kl}p_kq_l + \cdots + a_{mn}p_mq_n\\
&= \begin{pmatrix} p_1 & p_2 & \cdots & p_m \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}\\
&= \mathbf{p}A\mathbf{q}^{\mathrm{T}}
\end{aligned}
$$

Expected Payoff

Similarly the expected payoff of Player II is

$$E(P_B)$$

$$= \begin{pmatrix} p_1 & p_2 & \cdots & p_m \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

$$= \mathbf{p} B \mathbf{q}^T$$

Product and difference game

Each of the two players of a game chooses one number from “2” or “-1” simultaneously. Then the payoffs of Player I and Player II are the product and difference of the two numbers respectively.

Two-person Game

The game can be represented by the matrix

		Player II (Column Player)	
		2	-1
Player I (Row Player)	2	(4,0)	(-2,3)
	-1	(-2,3)	(1,0)

Two-person Game

The payoffs of the two players are represented by the two matrices.

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$$

Payoff matrix of Player I

$$B = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

Payoff matrix of Player II

Two-person Game

Suppose Player I uses strategy (0.8,0.2) and Player II uses strategy (0.6,0.4). The payoff of Player I is

$$E(P_A) = \begin{pmatrix} 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 1.6 \\ -0.8 \end{pmatrix}$$

$$= 1.12$$

Two-person Game

Suppose Player I uses strategy (0.8,0.2) and Player II uses strategy (0.6,0.4). The payoff of Player II is

$$E(P_B) = \begin{pmatrix} 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 2.4 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$= 1.32$$

Two-person Game

Suppose Player II uses (0.6,0.4), we may calculate

$$A\mathbf{q}^T = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$= \begin{pmatrix} 1.6 \\ -0.8 \end{pmatrix}$$

It shows that payoffs of Player I will be 1.6 and -0.8 if he plays “2” and “-1” respectively.

Two-person Game

Suppose Player I uses (0.8,0.2), we may calculate

$$\mathbf{p}B = \begin{pmatrix} 0.8 & 0.2 \end{pmatrix} \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.6 & 2.4 \end{pmatrix}$$

It shows that payoffs of Player II will be 0.6 and 2.4 if he plays "2" and "-1" respectively.